Recall:

Thm A: Noetherian, dim 1.

TFAE:

1. Unique factor of ideals
2. Locally PID
3. Integally closed.

Def: A satisfying (1)-(3) is called a Dedekind domain.

The main motivation: \text{Spec}(A) non-singular $\longleftrightarrow$ A Dedekind.
Some basic geometry of smooth curves.

Consider the function

\[ A'(C) = C \rightarrow C = A'(C) \]

\[ \mathbb{R} \rightarrow \mathbb{R}^n \]

\[ \text{Ex } (n=2) \]

Our the codomain, the fibers of this map look like:

...diagram...

fifteen have continuity n everywhere except at zero, where fifi is a single point. Another interesting bit of structure is mandatory:

...diagram...

Where "lifting" a loop in codomain results in a path from a to -a in the domain.
Or, here's another way to visualize:

This number $n$ (set in example) seems important. Generically, it's the size of the fiber. But what about at the origin? Is there a way to detect it?

Let $m = (x, y) \in \operatorname{MaxSpec}(C[x]) \hookrightarrow \text{origin of } A^1$.

The map $z \mapsto z^2$ is modeled by

$$C[[z]] \leftarrow C[[x,y]] / \left( y^2 - x \right) \leftarrow C[[x]]$$

$$f(x) \quad \leftarrow \quad 1 \quad f(x)$$

$$h(z) \leftarrow g(y^2)$$

$$g(y^2)$$

$$h(y)$$

i.e.,

$$f(z^2) \leftarrow f(x),$$

(Pulling back a polynomial $A^1 \leftarrow C$ means "substitute $x$ with $z^2$.")
So we have

\[
\begin{align*}
\mathbb{C}[x,y]/(y^2-x) & \longrightarrow L = \mathbb{C}[x]/(y^2-x) \\
\mathbb{C}[x] & \longrightarrow K = \mathbb{C}(x)
\end{align*}
\]

where \( K \subset L \) is a Galois extension of degree 2.

What does it mean to study the fiber of a map?

Given a point \( a \in \text{Spec} A \) (\( a \) a maximal ideal of \( A \)),

the fiber of \( a \) in \( \text{Spec} B \) is the ideal generated by \( a \) in \( B \).

That is, viewing \( A \subset B \), it makes sense to consider \( B_a \),

the ideal generated by \( a \).

\[
\begin{align*}
E_x = A &= \mathbb{C}[x] \\
&\longrightarrow \mathbb{C}[x]/(x^2 - x) \\
&\cong \mathbb{C}[z] \\
&\longrightarrow \mathbb{C}(z)
\end{align*}
\]

If \( m = (x-a), a \neq 0 \), then \( mB = (z^2 - a) = (z + 3a, z - 2a) \) in \( B \).

That is,

\[
mB = M_1, M_2
\]

for two maximal ideals of \( B \). (Such a factorization exists when \( B \) is Dedekind.)

If \( m = (x), a = 0 \), then \( mB = (z^2) = (z)^2 \). So \( mB = M^2 \) for \( M \) maximal.
So the number of factors (with multiplicity) in the prime ideal factorization of $mB$ seems to recover $2$ ($= n$).

More examples:

$A = \mathbb{Z}$  \hspace{1cm} The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$

$B = \mathbb{Z}[i]$. \hspace{1cm} is \hspace{1cm} $\mathbb{Z} \hookrightarrow \mathbb{Z}[i] \rightarrow \mathbb{Z}[i] / (x^2 + 1) \cong \mathbb{Z}[i].$

This gives rise to

$B = \mathbb{Z}[i] \hookrightarrow \mathbb{C}[i] = L$

$U \hookrightarrow U$

$A = \mathbb{Z} \hookrightarrow \mathbb{Q} = K$


$m = (2)$: $mB = (2) = (1+i)(1-i) = 2(1+i)^2 = (1+i)^2 = M^2.$

$m = (3)$: $mB = (3)$, which is prime. This does not recover "2." However, consider $B/(3)B$ vs $A/(3)$. Note we have $A/m \hookrightarrow B/M$, so we have a finite field extension.

$B/M = B/(3) \cong \mathbb{Z}_3 \cong \mathbb{Z}_3[x]/(x^2 + 1) \cong \mathbb{F}_3$, $A/M = A/(3) \cong \mathbb{Z}_3 \cong \mathbb{F}_3$.

$[B/m : A/m] = 2$! Recover $2$. 

$m = (5)$: Then $mB = 5B = (5) = (1 - 2i)(1 + 2i) = M_1M_2$. have two 
distinct primes in factorization.

It's hard to see from these examples, but here's the general pattern:

$\text{Def. Fix } B \rightarrow \text{ L}$

$U \quad U$

$A \rightarrow K$

where $[L:K] < \infty$, $B$ is an $A$-module, and

- $A, B$ Dedekind
- $K, L$ are fields of $A, B$, respectively

$B = \text{integral closure of } A \text{ in } L$.

(Turns out any two implies the third.) Then for any $m \in \text{MaxSpec}(A)$, write

$mB = M_1^{e_1} \cdots M_n^{e_n}$ prime factor of ideal in $B$

We say $e_i$ is the \textit{ramification degree} of $M_i$ over $m$.

Since $A/m \hookrightarrow B/M_i$ is a field extension,

$f_i := [B/M_i : A/m]$ 

is called the \textit{residual degree} of $M_i$ over $m$. 
Thm: \( m \in \text{MaxSpec} (A) \), as above,

\[ [L:K] = \sum_{i=1}^{\infty} e_i f_i. \]

Rem: We saw when \( A = \mathbb{Z}, \ B = \mathbb{Z}[\tau] \) that to compute \( f_i \), we needed to do computations mod \( p \), i.e., positive characteristic naturally appears.

Defn: \( M \) is ramified over \( A \) (or over \( m = m(A) \)) if
- \( e > 1 \), or
- \( A/m \to B/m \) is not separable.

Thm: A Dedekind if \( eA_{r,y} \) more reducible, and \( M \in \text{MaxSpec} (C_{f}) \). If \( C_{f} \) is a Dedekind domain, \( M \) is ramified over \( A \)
iff \( f'(y) \in M \) (i.e., \( f'(\alpha) = 0 \) in \( \mathbb{Q}/M \)).

Rem: \( A \to A_{r,y} \cong C_{f} \)

\( A/m \to C_{f}/M \in \mathfrak{m} \).

\( A_{r,y} = A_{r,y}(\mathbb{Q}/M) \) some part of \( \mathbb{Q}/M \).
Now, Grunb thry:

Thm If \( K \subset L \) finite Galois,

\[
\begin{array}{c}
B \\
\U \\
A \\
\end{array} \longrightarrow \begin{array}{c} K \\
\U \\
L \\
\end{array}
\]

as before, then

1. \( \not\exists \sigma \in \text{Gal}(L/K), \quad \sigma(B) = B. \)

2. \( \not\exists m \in \text{MaxSpec}(A), \quad \not\exists M, M' \text{ over } m, \exists \sigma \text{ st } \sigma M = M'. \)

3. \( e_M = e_{M'}, \quad f_M = f_{M'}, \quad \not\exists m, M' \text{ over } m, \text{ so } \)

\[mB = (M_1 \cdots M_K)^e\]

and \( [e : k] = [L : K]. \)

\( k \), ramification looks the same at all ramification points above a given fiber. Note that restricting the # of ramification points above a given \( m \).
Note: \( \sigma(B) = B \) means \( \text{Gal}(L/K) \) acts on \( B \), hence on \( \text{Spec}(B) \) (from left).

Moreover,

\[ (5) \quad A = B^G, \quad \text{and} \]

\[ \text{Spec}(B) / G \cong \text{Spec}(B^G) = \text{Spec}(A). \]

We finish with a more concrete application.

Let \( K = C(t) \).

If \( K \) is finite, let \( L = K(x) \). (Choose \( 0 \to 1 \) separable; apply prime off the and set \( f \) to be irreducible polynomial of \( x \):

\[ f \in K[t][x], \quad f(t, x) = a_n x^n + \cdots + a_0 \]

where we can choose \( a_i \in K[t] \) (by clearing denominators).

\[ g(t, a_1, \ldots, a_{n+1}) = 1, \]

\[ a_n \text{ monic in } t. \]

Then \( f \) defines \( Z_f \subset A^2 = \{ (t, x) | f(t, x) = 0 \} \), with \( Z_f \to A^1 \)

having branch point to \( \in A^1 \) if \( f(t_0, x) \) has repeated roots.
Then (Riemann Existence)

\[ \{ \text{isom classes of extensions} \} \quad K \subset L, \quad [L:K] = d \quad \cong \quad \{ \text{isom classes} \}
\]

\[ \text{of } \delta \text{-sheeted branched covers of } A' \]

\[ L \xrightarrow{\alpha} L' \]

\[ X \xrightarrow{\beta} X' \]


generically

1 sheet

\[
\begin{array}{c}
\text{multiple roots} \\
\text{finitely many branch points} \\
\end{array}
\]

When cover is Galois, all singularities above \( m \) are the same.

\[ \text{If } K \subset L \text{ Galois, over a point } m \in A', \text{ have that all } p \text{ ramified over } p \text{ "look the same," by previous theorem.} \]