Last time:

Thm 3. bijection

\[ \text{MaxSpec}(k[x_1, \ldots, x_n]) \cong k^n \]

when \( k \) alg. closed.

**Remark** This justifies two ideas at once: (1) \( k[x_1, \ldots, x_n] \) is a ring of functions on \( k^n \).

(2) Max ideals are "points."

If summary: Let \( R = k[x_1, \ldots, x_n] \).

(1) If \( m \subset R \) maximal, \( R/m \cong k \).

(2) \( \exists i, \phi_i: k[x_i] \hookrightarrow R, \phi_i^{-1}(m) \) is maximal.

(3) \( m = (x_1-a_1, \ldots, x_n-a_n) \) is maximal if \( (a_1, \ldots, a_n) \in k^n \).

Then we have \( k^n \cong \text{MaxSpec}, \) an injection \( \text{blc} R/m \cong k, \phi_i^{-1}(m) \cong k 

and \( x_i-a_i, x_i-a_i^2 \) to different elements if \( a_i \neq a_i^2 \).

**Conclusion by (3):** If \( m \subset R \) maximal, (2) tells us \( (x_i-a_i, x_i-a_i^2) \in \text{MaxSpec}. \)
Remark. Logically, (1) isn’t necessary. Apologies.

Recall we proved

Lemma \(\mathbb{I}\): \( R \) is field over \( \mathbb{K} \).

If \( \Theta \) inclusion of \( \mathbb{K} \)-algebras

\[ R \subseteq S \]

where \( S \) is fin-gen as \( \mathbb{K} \)-alg,

then every \( a \in R \) is algebraic.

Remark. Didn’t finish pf in class, but look @ notes.

Now let me prove

Lemma \(\mathbb{II}\): Let \( S \) be a \( \mathbb{K} \)-alg, finitely generated.

The \( \mathbb{K} \)-alg maps

\[ \phi: R \rightarrow S \]

\( \phi^{-1} \) sends max ideals to max ideals.
**Proof:** Consider $m \subseteq S$ maximal, and the inclusion

$$R'/k(m) \rightarrow S/m$$

In goal, $\Phi^{-1}(m)$ is prime (see Remark: $\Phi^{-1}(\text{primes}) = \text{primes } K$) so have a PID including into a field $S/m$.

Apply Lemma 1 to $S/m \rightarrow S/m$ is both "$R" and "$S" of hypothesis.

Then $a \in S/m \Rightarrow a$ is alg over $k$

$$\Rightarrow "a \in R'/k(m) \Rightarrow a$ is alg over $k'"$$

$$\Rightarrow "a \in R'/k(m) \Rightarrow a$ has mult more in $R'/k(m)"$$

$$\Rightarrow R'/k(m)$ is a field.

$$\Rightarrow \Phi^{-1}(m)$ is maximal. $\Box$

**Cor:** $S$ a field over $k$, fin-gen. as $k$-alg.

Then $S$ is fin-dim. over $k$. 

**(pf of (ii)).**

**pf:** By induction on $\#$ of generators of $S$.

$l=1$: Obvious as $S \cong k(\alpha)$, hence $S$ is fin-dim. over $k$.

$l \to l+1$: Note that given $\alpha_1, \ldots, \alpha_l$ generators of $S$,

$L := k(\alpha_1, \ldots, \alpha_l) \subseteq S$ is a fin-dim. ext of $k$ by induction.

Then $L \times k \rightarrow S$, $x \mapsto \alpha_{l+1}$ shows $S$ is finite extension over $L$. $\Box$

**Cor:** Assertion (2).
Lemma 7.7. \((a_1, \ldots, a_n) \in \mathbb{F}^n, (x_1-a_1, \ldots, x_n-a_n) \in m \subset R\)

\(B\) maximal.

\[\text{Proof: Consider ring map}\]

\[ev : R \rightarrow \mathbb{k}\]

\[f \mapsto f(a_1, \ldots, a_n).\]

\[\text{Composite}\]

\[\mathbb{k} \subset \text{polys} : R \rightarrow \mathbb{k}\]

is a ring isom, so \(ev\) surjection. Moreover, obvious that

\[f \sim f(a_1, \ldots, a_n) \mod m.\]

Since composition also shows \(a_0 \sim a_0' \mod m\) if \(a_0 = a_0'\), induced map \(ev'\) in

\[\mathbb{k} \rightarrow R/m \xrightarrow{ev'} \mathbb{k}\]

is an isom. So \(R/m \cong \text{field}.\)
Here's one of the original motivations for Nullstellensatz:

**Exercise:** \( f_1, \ldots, f_e \in R = \mathbb{K}[x_1, \ldots, x_n]. \)

Assume \( \exists g_1, \ldots, g_e \in R \) s.t.

\[
\sum_{i=1}^{e} f_i g_i = 1 \quad (\text{8})
\]

Show \( V(f_1, \ldots, f_e) = \{ (a_1, \ldots, a_n) \in \mathbb{K}^n \mid f_i(a_1, \ldots, a_n) = 0 \} \) is empty.

**Proof:** \( (\sum f_i g_i)(a_1, \ldots, a_n) = 1. \)

So each \( f_i \) could not evaluate to \( 0 \) at \( (a_1, \ldots, a_n) \).

\[\text{Then (Weak Nullstellensatz )} \quad \text{The converse holds} \]

That is, if \( I \neq R \) is an ideal, then

\[ V(I) \neq \emptyset. \]

**Remark:** If \( \exists g_i \) s.t. \( \sum f_i g_i = 1 \), \( I := (f_1, \ldots, f_e) = (1) \), \( R \).

The weak Nullstellensatz says the obvious obstacle (8) to the mutual vanishing of the \( f_i \) is the only obstacle.
The proof of the weak Nullstellensatz relies on Zorn's lemma.

**Lemma (Zorn) Let \((P, \leq)\) be a poset such that any chain has an upper bound. Then \(P\) has at least one maximal element.

**Rank Here,**

- A chain is a clux \(\{p \in P \mid p \leq \alpha, \beta\}\).
- \(p_\alpha \leq p_\beta \text{ or } p_\beta \leq p_\alpha\)
- An upper bound is any \(x \in P\) s.t. \(\forall \alpha, p_\alpha \leq x\).
- A maximal elt is any \(x\) s.t. \(x \leq y \Rightarrow x = y\).

**Rank Zorn's lemma is equivalent to the axiom of choice.**

This is the "C" in ZFC, and we definitely assume the axiom of choice in this class. We do not prove either implication:

- Zorn's lemma \(\iff\) Axiom of choice

in this class.
Prove let $R$ be a ring.
Then any ideal $I \subseteq R$ is contained in a maximal ideal.

Let $P$ be the poset

$P := \{\text{ideals } J \subseteq R \mid I \subseteq J, \quad 1 \not\in J^2\}$,

ordered by inclusion. Fix a chain $\{J_\alpha\}$. Then

$J := \bigcup \alpha J_\alpha$

is in $P$. ($x, x' \in J, J_\beta \Rightarrow x \in J_\beta$ wlog. Here

$x + x' \in J_\beta \subseteq J$.

$J$ is obviously a submodule.

So Zorn's lemma applies. //

(If Weak Nullstellensatz)

Let $m$ be a maximal ideal containing $I$.
Then $m = (x_1 - a_1, \ldots, x_n - a_n)
= \ker (ev_{a_1}, \ldots, ev_{a_n})
= \{ f \mid f(a_1, \ldots, a_n) = 0 \}$

so $V(I) \supseteq (a_1, \ldots, a_n) \subseteq \mathbb{k}$.