Lecture 17: Limits, colimits, and some rings

17.1. Colimits.

**Definition 17.1.** A diagram in \( C \) is a functor \( F : \mathcal{D} \to C \). We say \( F \) is a diagram of shape \( \mathcal{D} \).

**Example 17.2.** Let \( \mathcal{D} = * \coprod * \) be the category with two objects and only identity morphisms. A diagram in the shape of \( \mathcal{D} \) picks out two objects \( X, X' \) of \( C \).

**Definition 17.3.** Fix a category \( \mathcal{D} \). The (left) cone category on \( \mathcal{D} \), denoted \( \mathcal{D}^\circ \), is the category where

1. \( \text{Ob} \mathcal{D}^\circ := \text{Ob} \mathcal{D} \coprod \{ * \} \)
2. \( \text{hom}_{\mathcal{D}^\circ}(X, Y) := \begin{cases} \text{hom}_\mathcal{D}(X, Y) & X, Y \in \text{Ob} \mathcal{D} \\ pt & Y = * \\ \emptyset & \text{otherwise} \end{cases} \)

Note, for instance, that even if \( \mathcal{D} \) already has a terminal object, \( \mathcal{D}^\circ \) has a new terminal object, and it is not isomorphic to the original terminal object of \( \mathcal{D} \).

Note also that composition is forced upon you, as the only new morphism spaces are empty or are singletons.

**Example 17.4.** With \( \mathcal{D} \) as above, \( \mathcal{D}^\circ \) looks as follows:

\[
\begin{array}{c}
\ast \\
\downarrow \\
\ast \rightarrow \ast
\end{array}
\]
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Definition 17.5. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. Then define

$$\text{Fun}_D(\mathcal{D}^p, \mathcal{C})$$

to be the category where

1. An object is a functor $F' : \mathcal{D}^p \to \mathcal{C}$ such that $F'|_\mathcal{D} = F$; i.e., the restriction to $\mathcal{D}$ is the original diagram.
2. A morphism is a natural transformation $\eta : F' \to F''$ such that $\eta|_\mathcal{D} = \text{id}_F$; i.e., the restriction to $\mathcal{D}$ is just the identity natural transformation.

Remark 17.6. The notation does not indicate the dependence on $F$.

Example 17.7. Continuing the previous example, an object of $\text{Fun}_D(\mathcal{D}^p, \mathcal{C})$ picks out a diagram of the shape

$$\begin{array}{ccc}
X & \xrightarrow{f'} & Z \\
\downarrow{f'} & & \downarrow{g'} \\
X' & \xleftarrow{g'} & Z
\end{array}$$

and a morphism in this category picks out a commutative diagram as follows:

$$\begin{array}{ccc}
X & \xrightarrow{f} & W \\
\downarrow{f''} & & \downarrow{g''} \\
X' & \xleftarrow{g'} & Z
\end{array}$$

Definition 17.8. Fix a category $\mathcal{E}$. An initial object in $\mathcal{E}$ is an object $X$ such that $\text{hom}(X, Y) = pt$ for any $Y \in \mathcal{E}$.

Note any two initial objects are isomorphic.

Definition 17.9. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. A colimit for $F$ is an initial object of the category $\text{Fun}_D(\mathcal{D}^p, \mathcal{C})$. 
Example 17.10. Continuing the previous example, an initial object of \( \text{Fun}_D(D, \mathcal{C}) \) is some diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & X' \\
\downarrow & & \downarrow \quad g' \\
 & Z & \quad f'' \\

downarrow g'' & & \downarrow h \\
& \quad W & 
\end{array}
\]

such that for any other diagram (below indicated using \( f'', g'', W \)) there is a unique morphism \( h: Z \rightarrow W \) making the following commute:

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & X' \\
\downarrow f'' & & \downarrow g'' \\
& Z & \quad W \\

downarrow h & & \downarrow h \\
& \quad 
\end{array}
\]

Definition 17.11. A colimit in the shape of \( D = * \coprod * \) is called a coproduct in \( \mathcal{C} \).

17.2. Limits. One can likewise define limits as an initial object in

\[ \text{Fun}_{D^{\text{op}}}(D^{\text{op}}, \mathcal{C}^{\text{op}}). \]

But this is opaque. Here are the dual constructions to define limits, spelled out:

Definition 17.12. Fix a category \( D \). The (right) cone category on \( D \), denoted \( D^{\triangleleft} \)

is the category where

\[
\begin{align*}
(1) \quad \text{Ob } D^{\triangleleft} := & \{ * \} \coprod \text{Ob } D \\
(2) \quad \text{hom}_{D^{\triangleleft}}(X, Y) := & \begin{cases} 
\text{hom}_D(X, Y) & X, Y \in \text{Ob } D \\
p t & X = * \\
\emptyset & \text{otherwise}
\end{cases}
\end{align*}
\]

Example 17.13. With \( D \) as above, \( D^{\triangleleft} \) looks as follows:

\[
\begin{array}{ccc}
* & \xrightarrow{*} & * \\
\downarrow & & \downarrow \\
* & & *
\end{array}
\]
Definition 17.14. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. Then define
\[
\text{Fun}_\mathcal{D}(\mathcal{D}^\downarrow, \mathcal{C})
\]
to be the category where
\begin{enumerate}
\item An object is a functor $F' : \mathcal{D}^\downarrow \to \mathcal{C}$ such that $F'|_\mathcal{D} = F$; i.e., the restriction to $\mathcal{D}$ is the original diagram.
\item A morphism is a natural transformation $\eta : F' \to F''$ such that $\eta_\mathcal{D} = \text{id}_F$; i.e., the restriction to $\mathcal{D}$ is just the identity natural transformation.
\end{enumerate}

Definition 17.15. Fix a category $\mathcal{E}$. A terminal object in $\mathcal{E}$ is an object $Y$ such that $\text{hom}(X, Y) = \text{pt}$ for any $X \in \mathcal{E}$.

Definition 17.16. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. A limit for $F$ is a terminal object of the category $\text{Fun}_\mathcal{D}(\mathcal{D}^\downarrow, \mathcal{C})$.

Table 3. Some notation. Note that the word “limit” can be used to describe both limits and colimits in the literature; the words “inverse” or “directed” give indication of whether one’s talking about limits or colimits.

<table>
<thead>
<tr>
<th>Colimits</th>
<th>Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{colim}(F : \mathcal{D} \to \mathcal{C})$</td>
<td>$\text{lim}(F : \mathcal{D} \to \mathcal{C})$</td>
</tr>
<tr>
<td>$\text{colim}_\mathcal{D} F$</td>
<td>$\text{lim}_\mathcal{D} F$</td>
</tr>
<tr>
<td>$\text{colim} F$</td>
<td>$\text{lim} F$</td>
</tr>
<tr>
<td>$\text{lim}_\to \mathcal{D}$</td>
<td>$\text{lim}_{\to} \mathcal{D}$</td>
</tr>
<tr>
<td>“directed limit”</td>
<td>“inverse limit”</td>
</tr>
</tbody>
</table>

17.3. Exercises.

Exercise 17.17. Articulate the universal property of quotients of $R$-modules using colimits.

Exercise 17.18. Articulate the $p$-adic integers as a limit in rings.

Solutions: Fix $f : A \to B$ a map of $R$-modules and $\pi : B \to B/f(A)$ the quotient map. The universal property of quotients $B/f(A)$ says that for any map $\phi : B \to C$ of $R$-modules for which $\ker \phi \supset f(A)$, there is a unique morphism $\phi' : B/f(A) \to C$ such that $\phi' \circ \pi = \phi$. 

But the requirement \( \ker \phi \supset f(A) \) is the expressing the commutativity of the following diagram:

\[
\begin{array}{c}
A \\
\downarrow \downarrow
\end{array}
\begin{array}{c}
B \\
\downarrow \phi
\end{array}
\]

And the universal property is expressing the uniqueness of \( \phi' \) in the commutative diagram below:

\[
\begin{array}{c}
A \\
\downarrow \downarrow
\end{array}
\begin{array}{c}
B \\
\downarrow \phi
\end{array}
\]

So \( \mathcal{D} = \ast \leftarrow \ast \rightarrow \ast \) is the shape, and any functor \( \mathcal{D} \to RMod \) looking like

\[
\begin{array}{c}
A \\
\downarrow \\
0
\end{array}
\begin{array}{c}
B \\
\downarrow \phi
\end{array}
\]

has a colimit given by the quotient \( B/f(A) \) (equipped with the quotient map \( B \to B/f(A) \)).

As for the next exercise, the \( p \)-adics can be written as a limit

\[
\ldots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}
\]

of a functor from \( \mathcal{D} = (\mathbb{Z}_{\leq 0}, \leq) \) to Rings.

Analogously, we have the sequence of rings

\[
\ldots \to \mathbb{C}[x]/x^3 \to \mathbb{C}[x]/x^2 \to \mathbb{C}[x]/x \cong \mathbb{C}
\]

whose limit is \( \mathbb{C}[[x]] \), the ring of power series. This sequence has a geometric interpretation: \( \mathbb{C}[x]/x \cong \mathbb{C} \) is the functions on the origin in \( \mathbb{A}^1 \) (the complex line), and \( \mathbb{C}[x]/x^n \) is the \((n - 1)\)st order neighborhood.