Lecture 15: Fun example: Cobordisms.

Most of what we covered is in the TeXed notes for Lecture 14. However, I did cover one new topic: Cobordism categories.

So far, all the categories we’ve studied look like “sets” underlie them. In particular, in \( C = \text{groups}, \text{rings}, \text{vector spaces}, \text{sets} \), all the morphisms \( f : X \to Y \) are always functions satisfying some property.

Well, here is an example where the category’s morphisms are not of that flavor.

**Definition 15.1.** Let \( \text{Cob}_1 \) be the category of oriented, 1-dimensional cobordisms.

1. An object \( X \) is given by a finite subset of \( \mathbb{R}^\infty \), each element given a sign of plus or minus. One should think of this as a collection of points floating in space, each point with an orientation.

2. A morphism from \( X_0 \) to \( X_1 \) is the data of an subset \( \Gamma \subset \mathbb{R}^\infty \times [0, 1] \), with the data of an orientation, satisfying the following conditions:
   - \( \Gamma \) is a disjoint union of smooth curves, possibly with boundary. We demand that the boundary of \( \Gamma \) is precisely the intersection of \( \Gamma \) with \( \mathbb{R}^\infty \times \{0\} \cup \mathbb{R}^\infty \times \{1\} \).
   - We demand that the boundary of \( \Gamma \) at 0 is precisely \( X_0 \), and the boundary of \( \Gamma \) at 1 is precisely \( X_1 \). These must be compatible with the orientations.
   - We declare two \( \Gamma \) to be equal if one can be smoothly transformed (isotoped) into the other while respecting boundaries.

3. Composition: If \( \Gamma : X_0 \to X_1 \) and \( \Gamma' : X_1 \to X_2 \), the composition is given by gluing \( \Gamma \) and \( \Gamma' \) along \( X_1 \). Note that this naturally lives over the interval \([0, 2]\), but we can reparametrize this interval. This is compatible with the isotopy equivalence relation above.
Some examples of objects: The empty subset, a singleton subset with positive orientation, a singleton subset with negative orientation, and disjoint unions of these.

Some examples of morphisms: $\Gamma$ could be

1. A circle, which has no boundary; this is a morphism from $\emptyset$ to $\emptyset$.

2. A horseshoe, with boundary only on $\mathbb{R}^\infty \times \{0\}$. Necessarily, the boundary will consist of a $+$ point and a $-$ point.

3. A co-horseshoe, with boundary only on $\mathbb{R}^\infty \times \{1\}$. Necessarily, the boundary will consist of a $+$ point and a $-$ point.

4. A single line interval with one boundary point on $\mathbb{R}^\infty \times \{0\}$ and the other on $\mathbb{R}^\infty \times \{1\}$. Necessarily, these two boundary points have the same orientation; this is the identity morphism from the boundary point to itself.

**Theorem 15.2.** Let $Z : \text{Cob}_{1}^{\infty} \to \text{Vect}_k$ be a functor taking $\amalg$ to $\otimes_k$. Then $Z(*_+) := V_+$ is finite dimensional, and one can identify $Z(*_-)$ with its dual.

We'll articulate this more accurately next lecture.