Lecture 7

Notes from class may differ from written notes.

Last time, stated:

\textbf{Lemma 1}\hspace{1em} \text{End}_R(k) \cong R^{op}

\textbf{Lemma 2}\hspace{1em} \text{End}_k(V_{reg}) \cong \bigoplus \text{M}_{n_x \times n_y}(C)

where \{f\} runs over irreps \(V_x\) of \(G\),
\[ n_x = \dim(V_x) \]

\textbf{Lemma 3}\hspace{1em} Z(CG_n) = C(G_n) = \{\text{class funs}\}

Showed Lemmas \implies C(G_n) \cong \frac{k[G]}{G_n}

\implies \# \text{irreps} = \frac{\# k}{\# G_n}.

Also proved Lemma 1.

Today: prove Lemmas 2, 3.
Def. Fix a clan of vector spaces $\{V_\alpha\}$ where $\mathcal{A} := \mathcal{A}$. Then the direct product of the $V_\alpha$ is defined to be the vector space

$$\prod_{\alpha \in \mathcal{A}} V_\alpha := \{ (v_\alpha)_{\alpha \in \mathcal{A}} \mid v_\alpha \in V_\alpha \}$$

The direct sum of $\{V_\alpha\}$ is the vector space

$$\bigoplus_{\alpha \in \mathcal{A}} V_\alpha \subset \prod_{\alpha \in \mathcal{A}} V_\alpha$$

of those $(v_\alpha)$ s.t. only finitely many $v_\alpha$ are non-zero.

Remark: When $\mathcal{A}$ finite,

$$\bigoplus_{\alpha \in \mathcal{A}} V_\alpha \cong \prod_{\alpha \in \mathcal{A}} V_\alpha$$

is an isomorphism.

Note that there are projections

$$\prod_{\alpha \in \mathcal{A}} V_\alpha \xrightarrow{P_\alpha} V_\alpha, \quad (v_\alpha) \mapsto v_\alpha$$

and inclusions

$$V_\alpha \hookrightarrow \bigoplus_{\alpha \in \mathcal{A}} V_\alpha, \quad v_\alpha \mapsto (v_\beta)$$

where $V_\beta := \begin{cases} V_\alpha & \alpha \neq \beta \\ \{0\} & \text{otherwise} \end{cases}$
The projections and inclusions induce maps (for any $W$)

\[
\begin{align*}
\text{hom}(W, \prod V_d) & \to \text{hom}(W, V_d) \\
\text{f} & \mapsto \rho_a \circ \text{f} \\
\text{hom}(\bigoplus V_d, W) & \to \text{hom}(V_d, W) \\
\text{g} & \mapsto g \circ \iota_a
\end{align*}
\]

hence maps

\[
\begin{align*}
\text{hom}(W, \prod V_d) & \to \prod_d \text{hom}(W, V_d) \\
\text{f} & \mapsto (\rho_a \circ \text{f})_d \\
\text{hom}(\bigoplus V_d, W) & \to \prod_d \text{hom}(V_d, W) \\
\text{g} & \mapsto (g \circ \iota_a)_d
\end{align*}
\]

these maps are both isomorphisms, and natural. In English, these isomorphisms mean:

"To give maps $\{f_a: W \to V_d\}$

is the same as giving a single map $f: W \to \prod V_d$.

"To give maps $\{g_d: V_d \to W\}$

is the same as giving a single map $g: \bigoplus V_d \to W$."
Now suppose \( A = \mathbf{E} d \mathbf{3} \) is finite, so \( \bigoplus V_\beta = \mathbf{T} V_\alpha \).

Then we have

\[
\hom \left( \bigoplus V_\beta, \mathbf{T} V_\alpha \right) = \hom \left( \bigoplus V_\alpha, \bigoplus V_\beta \right) =: \text{End} \left( V_\alpha \right)
\]

writing \( A = \mathbf{E} d \mathbf{3} = \{ \beta \} \) for sanity's sake.

But \( \text{End}(\text{anything}) \) is always a ring because endomorphisms compose. How do we read off the ring structure in \( \mathbf{T} \hom (V_\alpha, V_\beta) \)?

By naturality, composition is given as follows: if \( (\varphi_\alpha, \varphi_\beta) \in \mathbf{T} \hom (V_\alpha, V_\beta) \),

\[
(\varphi_\beta)_{\alpha, \beta} \in \mathbf{T} \hom (V_\beta, V_\gamma),
\]

then

\[
(\psi_\beta)_{\alpha, \beta} \circ (\varphi_\beta)_{\alpha, \beta} = \left( \sum_{\beta} \psi_\beta \circ \varphi_\beta \right)_{\alpha, \beta} \in \mathbf{T} \hom (V_\alpha, V_\gamma).
\]

(Summation is finite by assumption on \( A \).)

"The \((\alpha, \gamma)\) entry is the sum of \( \psi_\beta \circ \varphi_\beta \) over all \( \beta \)."

This is just a generalized version of a matrix ring.
Now:

If (of Lemma 2)
\[ \text{End} (\oplus V_\alpha, \oplus V_\beta) \cong \prod_{\alpha \neq \beta} \text{hom}(V_\alpha, V_\beta) \]  
univ prop. (discussion so far)

\[ \cong \prod_{\alpha \neq \beta} \text{hom}(V_\alpha, V_\beta) \times \prod_{\alpha = \beta} \{0\} \]

Schur's Lemma

\[ \cong \prod_\alpha \text{End}(V_\alpha^{n_\alpha}) \]

\[ \cong \prod_\alpha \left( \text{hom}(\bigoplus_{i=1}^{n_\alpha} V_{i \beta}^{n_\beta}, \bigoplus_{j=1}^{n_\alpha} V_{i \beta}^{n_\beta}) \right) \]

\[ \cong \prod_\alpha \left( \prod_{i,j} \text{hom}(V_{i \beta}, V_{j \beta}) \right) \]

Schur's Lemma

\[ \cong \prod_\alpha \left( \prod_{i,j} \mathbb{C} \right) \]

since by naturality,
\[ (b_{i j} k_{a i j})_{i \alpha} = \sum_j b_{i j} k_{a i j}. \]
To prove Lemma 3, observe:

Prop: Fix unital ring \( R \) and \( r \in R \).

TFAE:

(i) \( \Phi_r: R \to R \) is a map of left \( R \)-modules

\[
\begin{align*}
\Phi_r(1) & = r \\
\Phi_r(x) & = rx
\end{align*}
\]

(ii) For every left \( R \)-module \( M \),

\[
\Phi_r: M \to M \\
\Phi_r(m) = rm
\]

is a map of left \( R \)-modules

(iii) \( \forall x \in R, \ x_r = rx \).

\( \text{Rank } (iii) \) means \( r \in Z(R) \),

i.e., \( r \) is in the center of \( R \).

\( \text{Hence, } (iii) \Rightarrow (i): \)

\[
\Phi_r(xm) = (rx)m = (rx)(rm) = x(rm) = x\Phi_r(m).
\]

(ii) \( \Rightarrow \) (i): obvious.

(i) \( \Rightarrow \) (iii): \( \Phi_r \) module map \( \Rightarrow \)

\[
\Phi_r(1) = \Phi_r(x \cdot 1). \text{ i.e., } x_r = rx. \]

\( \text{Thus, something "natural" about } \)

descriptor (i): it defines a map for:

any \( M \), and moreover, for any

morphism \( \psi: M \to M \), can check

\[
\begin{array}{cccc}
M & \xrightarrow{\Phi_r} & M \\
\downarrow{\psi} & & \downarrow{\psi} \\
M & \xrightarrow{\Phi_r} & M
\end{array}
\]

commutes. This leads to a notion of "center" for any category.
Now assume a \( f : G \rightarrow \mathbb{C} \)
\[
f: G \rightarrow \mathbb{C} \\
g \rightarrow f(g)
\]
(i.e., an element \( \sum_g f(g)e_g \in \mathbb{C}G \))
defines a module map of \( V_{reg} = \mathbb{C}G \).
Then have:
\[
(\sum_g f(g)e_g)(e_h) = h(\sum_g f(g)e_g).
\]
\( \forall e_h \in \mathbb{C}G \)
\[
\sum_g f(g)e_{gh} = \sum_g f(g)e_{hg}.
\]
Verify by \( h^{-1} \):
\[
\sum f_{gh}e_g = \sum f_{gh}e_{h^{-1}}.
\]