AMATH 352: Problem set 3

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1 Matlab Portion

1.1 Investigate rules going into LU decomposition

For the following matrix: $\mathbf{a} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 4 & 1 & 2 & 1 \end{bmatrix}$

(a) The operations required to perform Gaussian Elimination on this matrix can be expressed as matrix multiplication with a series of "unit" lower diagonal matrices. $L_3^{-1}L_2^{-1}L_1^{-1}a = U$. Each L_i^{-1} matrix zeros out one column below the diagonal. Define L_3^{-1}, L_2^{-1}, L_1^{-1} in your Matlab script and save them as `A11L3inv.dat`,`A11L2inv.dat`, and `A11L1inv.dat`. The command `eye(n)`, which produces and nxn identity matrix may be useful. Check that each matrix does what you expect.

(b) Calculate $L^{-1} = L_3^{-1}L_2^{-1}L_1^{-1}$. Save as `A11Linv.dat`. Note the structure of L^{-1}. Think about the following (Quiz question): Does it have a simple relationship with the Gaussian elimination factors?

(c) Define L_1, L_2, and L_3 by looking at L_1^{-1}, L_2^{-1}, L_3^{-1} and using the rules of inverse on unit lower diagonal matrices. You can check them using `Li = inv(Li_inv)` command. Save them as `A11L3.dat`,`A11L2.dat`, and `A11L1.dat`.

(d) Calculate L using the lower diagonal "unit" matrices. Save as `A11L.dat`. Compare the factors you calculated for gaussian elimination and the entries in L.

Summary of output for this section:

- `A11L3inv.dat`,
- `A11L2inv.dat`,
- `A11L1inv.dat`,

- A11Linv.dat,
- A11L3.dat,
- A11L2.dat,
- A11L1.dat,
- A11L.dat

1.2 LU decomposition algorithm

Starting with the Gaussian elimination (called `GE_NEW.m`) code I have provided you for this assignment, store the factors, and construct the lower diagonal matrix. Note that the way the row operations and factors, (`fac`), are defined is not the same as the Gaussian Elimination code in the previous assignment.

Change the code (and save with new filename) to be a new function `[L, U] = GE_LU(a,filename)`, output L and U. You can test your code by checking if A=LU. Moler's textbook online may have useful info for coding, if you are lost.

(a) $\mathbf{A12a} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 4 & 1 & 2 & 1 \end{bmatrix}$

(b) $\mathbf{A12b} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

(c) $\mathbf{A12c} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 5 & 11 \\ 12 & 10 & 15 & -13 \\ 8 & 6 & 12 & -1 \end{bmatrix}$

For the following test matrices save the partially constructed L and U to 'A12aL2.dat' and 'A12aU2.dat' files after zeroing out the 2nd pivot for the following matrices. Notice I have changed the save code inside the function to take the filename you pass into the function and add 'U2.dat' and 'L2.dat' onto the end of it to make the save names for the .dat files. You will need to define `filename = 'A12a'`, for example, outside your function and pass it to the function.

Summary of output for this section:

- `A12aL2.dat` and `A12aU2.dat`
- `A12bL2.dat` and `A12bU2.dat`
- `A12cL2.dat` and `A12cU2.dat`

1.3 Inverses, LU decomposition, and determinants

Recall in Lecture 6, we found how the we could define an determinant of a matrix to check if a matrix was nonsingular (and had an inverse), or singular (and no inverse existed).

(a) Use the matlab command `det(A12c)` to calculate the determinant for matrix from 1.2c. Save as 'A12cdet.dat'.

(b) Calculate the determinants of the L and U matrices that make up the LU decomposition of matrix A12c. Save as 'A12cLdet.dat' and 'A12cUdet.dat'.

(c) Calculate the determinant for the following matrix $\mathbf{A13c} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Save as 'A13cdet.dat'

(d) Examine the output for a-c, and think about the following questions for the quiz:
i) How could you rapidly calculate the determinant from the LU decomposition?
ii) How many solutions would there be to a set of equations defined by $\mathbf{A13cx} = \mathbf{b}$ where $\mathbf{b}$ is a non-zero vector? what if $\mathbf{b}$ is a zero vector?
ii) How does the number of solution of a system of equations relate to the determinant?

Summary of output for this section:

- `A12cdet.dat`
- `A12cLdet.dat` and `A12cUdet.dat`
- `A13cdet.dat`

2 Written portion

2.1 Linear Spaces

For each of the following, show that it is or is not a vector space. If it is, you must demonstrate all 10 conditions. If it is not, you only need to show that one condition fails.

(a) Under usual Matrix operations: $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ for } a, b \in \mathbb{R} \right\}$

(b) $W = \left\{ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \right\}$

2.2 Subspaces

(a) Which of the following sets W are subspaces of V? Justify your answers with an argument of why it is closed under addition and scalar multiplication or a counterexample showing it is not.

i $V = \mathbb{R}^3$ and $W = \left\{ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 \geq 0 \text{ and } x_3 = 0 \right\}$

ii $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$.

2.3 Linear Dependence and Independence

Recall the definition of linear independence:
The column vectors of matrix $\mathbf{S} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_k]$ are linearly independent if the only time any linear combination of them

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0$$

is when $c_1 = c_2 = \cdots c_k = 0$. If you can find coefficients $c_j \neq 0$ then the columns of $\mathbf{S}$ are linearly dependent.

Decide whether each subset is linearly dependent or independent. Justify your answer with a logical explanation. You can use the MATLAB command `rref(A)` to get the reduced row echelon form of the matrix `A` (the matrix after you have done Gaussian elimination operations until the system is as close to the inverse as possible).

(a) $\{ \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 14 \end{bmatrix} \}$ subset of $\mathbb{R}^3$

(b) $\{ \begin{bmatrix} 1 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix} \}$ subset of $\mathbb{R}^3$

(c) $\{ \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \}$ subset of $\mathbb{R}^3$

(d) $\{-x^2, 1 + 4x^2\}$ subset of $\mathbb{P}$,

(e) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$ subset of $\mathbb{P}$,